BERNSTEIN CENTER AND SCHOLZE'S BASE CHANGE

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1. Bernstein center

We review the basics of the Bernstein center as given in Chapter 2 of [Sch10] with some additional material from [Ber84]. Note that Bernstein uses G to be any reductive group over F.

Notation

- $\mathsf{F} :=$ a non-archimedean local field.
- $\mathsf{G} := \mathrm{GL}_{\mathsf{n}}(\mathsf{F})$.
- $\mathsf{K} :=$ a compact open subgroup of G.
- $e_{\mathsf{K}} := \frac{\chi_{\mathsf{K}}}{\mathrm{vol}(\mathsf{K})}$, the characteristic function of K divided by volume of K (idempotent associated to K).
- $\hat{\mathsf{G}} :=$ the set of irreducible smooth representations of G over $\mathbb{C}$.

The Hecke algebra $\mathcal{H}(\mathsf{G})$ of G is the convolution algebra of locally constant $\mathbb{C}$-valued functions on G with constant support. The Hecke algebra $\mathcal{H}(\mathsf{G}, \mathsf{K})$ consist of $\mathsf{f} \in \mathcal{H}(\mathsf{G})$ that are bi-invariant under K.

Fact 1.1.

$$\mathcal{H}(\mathsf{G}) = \varinjlim_{\mathsf{K}} \mathcal{H}(\mathsf{G}, \mathsf{K}).$$

Definition 1.2. We can define $\mathcal{Z}(\mathsf{G})$ abstractly as the endomorphism ring of the identity functor of the category of smooth complex representations of G.

Date: February 12, 2019.

Then $\mathcal{Z}(\mathsf{G})$ acts on any smooth representation and this action commutes with any G-morphism.

Goal 1.3. Describe the Bernstein center $\mathcal{Z}(\mathsf{G})$.

Definition 1.4. Denote the center of $\mathcal{H}(\mathsf{G},\mathsf{K})$ by $\mathcal{Z}(\mathsf{G},\mathsf{K})$ and let

$$\hat{\mathcal{H}}(\mathsf{G}) := \varprojlim_{\mathsf{K}} \mathcal{H}(\mathsf{G},\mathsf{K})$$

$$\mathcal{Z}(\mathsf{G}) := \varprojlim_{\mathsf{K}} \mathcal{Z}(\mathsf{G},\mathsf{K}),$$

where transition maps are given by applying idempotents (i.e. $\mathsf{f} \in \mathcal{H}(\mathsf{G},\mathsf{K}) \mapsto \mathsf{e}_{\mathsf{K}'} * \mathsf{f} * \mathsf{e}_{\mathsf{K}'}$ for $\mathsf{K}' \subset \mathsf{K}$). $\mathcal{Z}(\mathsf{G})$ is the *Bernstein center*.

Remark 1.5. To move from the abstract definition to the projective limit definition, the action of $\mathcal{Z}(\mathsf{G})$ on the permutation representation $\mathbb{C}[\mathsf{G}/\mathsf{K}]$ gives a morphism to $\mathsf{Z}(\mathsf{G},\mathsf{K})$.

Remark 1.6. In fact, $\mathcal{Z}(\mathsf{G}) \not\subset \mathcal{H}(\mathsf{G})$, but

$$\mathcal{Z}(\mathsf{G}) \subset \hat{\mathcal{H}}(\mathsf{G}) \supset \mathcal{H}(\mathsf{G}).$$

Using $\mathcal{H}(\mathsf{G}) = \varinjlim_{\mathsf{K}} \mathcal{H}(\mathsf{G},\mathsf{K})$, we know

$$\mathcal{H}(\mathsf{G})^\vee = \varprojlim_{\mathsf{K}} \mathcal{H}(\mathsf{G},\mathsf{K})^\vee \supset \varprojlim_{\mathsf{K}} \mathcal{H}(\mathsf{G},\mathsf{K}) = \hat{\mathcal{H}}(\mathsf{G}),$$

so we can define $\langle \mathsf{f}, \{\phi_{\mathsf{K}}\}_{\mathsf{K}} \rangle$ for any $\mathsf{f} \in \mathcal{H}\mathsf{G}$ and $\{\phi_{\mathsf{K}}\}_{\mathsf{K}} \in \hat{\mathcal{H}}(\mathsf{G})$, and therefore identify $\hat{\mathcal{H}}(\mathsf{G})$ with the space of distributions T of G such that $\mathsf{T} * \mathsf{e}_{\mathsf{K}}$ is of compact support for all compact open subgroups K (after choosing a Haar measure).

In fact, $\mathcal{Z}(\mathsf{G})$ is the center of $\hat{\mathcal{H}}(\mathsf{G})$ and is the space of such distributions that are conjugation-invariant.

Fact 1.7. $\hat{\mathcal{H}}(\mathsf{G})$ *has an algebra structure through convolutions of distributions and has center* $\mathcal{Z}(\mathsf{G})$*, which consists of the conjugation-invariant distributions in* $\mathcal{H}(\mathsf{G})$ *("geometrical realization of the Bernstein center").*

By Schur's lemma, we have a map ($\hat{\mathsf{G}}$ = irred. smooth repns of G).

$$\phi : \mathcal{Z}(\mathsf{G}) \to \mathrm{Map}(\hat{\mathsf{G}}, \mathbb{C}^\times)$$
$$z \mapsto \omega_{(\cdot)}(z)$$

giving an action of $\mathcal{Z}(\mathsf{G})$ on irreducible smooth representations of G via the central character. We may alternatively describe this as $\mathcal{Z}(\mathsf{G})$ acting on π via the "infinitesimal character" $\omega_\pi : \mathcal{Z}(\mathsf{G}) \to \mathbb{C}$ of π.

Definition 1.8. Let P be a parabolic subgroup of G with Levi subgroup $\mathsf{L} \cong \prod_{i=1}^k \mathrm{GL}_{n_i}$ and fix a supercuspidal representation σ of L.

Let $\mathsf{D} = \mathbb{G}_m^k$. Then we have a universal unramified character

$$\chi : \mathsf{L} \to \Gamma(\mathsf{D}, \mathcal{O}_\mathsf{D}) \cong \mathbb{C}[T_1^{\pm 1}, \dots, T_k^{\pm 1}]$$
$$(g_i) \mapsto \prod_{i=1}^k T_i^{\nu_p(\det(g_i))}.$$

We get a corresponding family of representations n–$\mathrm{Ind}_\mathsf{P}^\mathsf{G}(\sigma_\chi)$ (normalized induction) of G parametrized by D. We also write D for the set of representations of G obtained by specializing to a closed point of D (In Bernstein's language, D is a connected component of the set of isomorphism classes of irreducible cuspidal representations of L).

Let $\mathrm{Rep}\,\mathsf{G}$ be the category of smooth admissible representations of G and let $(\mathrm{Rep}\,\mathsf{G})(\mathsf{L}, \mathsf{D})$ be the full subcategory of $\mathrm{Rep}\,\mathsf{G}$ consisting of those representations that can be embedded into a direct sum of representations in D.

Theorem 1.9 ([Ber84, Proposition 2.10])**.** *As categories,*

$$\operatorname{Rep} \mathsf{G} = \bigoplus_{(\mathsf{L},\mathsf{D}) \textit{ up to conj.}} (\operatorname{Rep} \mathsf{G})(\mathsf{L}, \mathsf{D}).$$

This is a decomposition into a product of indecomposable abelian subcategories ("blocks").

Example 1.10. For a compact G,

$$\operatorname{Rep} \mathsf{G} = \bigoplus_{\hat{\mathsf{G}}} \mathbb{C}.$$

Definition 1.11. Let $W(\mathsf{L}, \mathsf{D})$ be the subgroup of $\mathsf{N}_\mathsf{G}(\mathsf{L})/\mathsf{L}$ ($\mathsf{N}_\mathsf{G}(\mathsf{L})$ is the normalizer of L in G) consisting of the n such that D (as a set of representations) coincides with its conjugate via n.

Remark 1.12. $W(\mathsf{L}, \mathsf{D})$ is a finite group acting on D.

Theorem 1.13 ([Ber84, Theorem 2.13])**.** *For any $z \in \mathcal{Z}(\mathsf{G})$, z acts by a scalar $c_{z,\pi}$ on $\pi := \text{n–Ind}_\mathsf{P}^\mathsf{G}(\sigma\chi_0)$ for any character χ_0. The corresponding function on D is a $W(\mathsf{L}, \mathsf{D})$-invariant regular function, inducing an isomorphism*

$$\mathcal{Z}(\mathsf{G}) \cong \textit{algebra of regular functions on } \bigcup_{(\mathsf{L},\mathsf{D})} \mathsf{D}/W(\mathsf{L}, \mathsf{D})$$

("spectral realization of the Bernstein center").

Remark 1.14. Theorem 1.13 is independent of P. We also really should replace D by $\mathsf{D}/\operatorname{Stab}(\sigma)$, quotienting out the σ-invariant elements of D.

Proof. Page 21 for proving first two sentences. Using the decomposition of Theorem 1.9, one only needs to show that $z \mapsto c_{z,\pi}$ identifies the center of the abelian category $(\operatorname{Rep} \mathsf{G})(\mathsf{L}, \mathsf{D})$ with the ring of regular functions on $\mathsf{D}/W(\mathsf{L}, \mathsf{D})$. Injectivity follows from every irreducible of $(\operatorname{Rep} \mathsf{G})(\mathsf{L}, \mathsf{D})$ being a subquotient of some $\text{n–Ind}_P^G(\tau)$ for some $\tau \in D$. Surjectivity follows from Lemma 1.15. $\square$

Lemma 1.15. *For every element* z_L *in the center of* $(\text{Rep } L)(D)$ *invariant with respect to* $W(L, D)$*, there exists an element* z *in the center of* $(\text{Rep } G)(L, D)$ *which acts on every induced representation* $\text{n–Ind}_P^G(\tau)$ *for* $\tau \in (\text{Rep } L)(D)$ *like* $\text{n–Ind}_P^G(\rho)$ *where* ρ *is the endomorphism of* τ *defined by* z_L*.*

Proof. For each $W \in (\text{Rep } G)(L, D)$ there is an injection

$$\varphi : W \hookrightarrow \bigoplus_P \text{n–Ind}_P^G \left((\text{Res}\,_P^G W)(D) \right).$$

The endomorphism $\bigoplus_P \text{n–Ind}_P^G(z_L)$ induces a functorial endomorphism on W, which is the element z that we want. Then it reduces to showing that the image $\varphi(W)$ is stable under $\bigoplus_P \text{n–Ind}_P^G(z_L)$. $\square$

Example 1.16 ([Ber84, Example 2.18])**.** Let $G = \text{GL}_n(H)$ where H is a division algebra over F. The conjugation classes of parabolic subgroups correspond to ordered partitions of n, the Levi subgroups correspond to $(n_i)_{1 \le i \le m}$ in the product of $\text{GL}_{n_i}(H)$ and $N_G(L)/L$ corresponds to the subgroup of S_m preserving the function $i \mapsto n_i$, which is a product of symmetric groups.

Let π_i be an isomorphism class of cuspidal representations of $\text{GL}_{n_i}(H)$ and let D_i be the orbit of π_i under twisting by an unramified character $\chi_T : g_i \mapsto T^{\nu_p(\det(g_i))}$. If F_i is the subgroup of the $(n_i\sqrt{[H:F]})$-th roots of unity such that $\pi_1 \cong \pi_i \chi_T$, then $D_i \cong \mathbb{C}^\times / F_i^\times$. Here, F_i corresponds to pieces of the Stabσ.

Let $D = \prod_i D_i$. Then $W(L, D)$ is the subgroup of S_m preserving $i \mapsto (n_i, D_i)$ and $D/W(L, D)$ is a nonsingular algebraic variety.

Example 1.17 ([Ber84, Example 2.19])**.** Let $G = \text{SL}_n(F)$, $L = \{$ diagonal matrices $\}$, and $D = \{\text{characters of } L/L^\circ\}$. Then $D \cong (\mathbb{C}^\times)^n / \Delta \mathbb{C}^\times$ and $W(L, D)$ is the permutation group of the indices.

The image of $(1, \zeta, \dots, \zeta^{n-1})$ in D, where ζ is a primitive n-th root of unity, has stabilizer $\langle(1, 2, \dots, n)\rangle$ in $W(L, D) \cong S_n$. For $n > 2$, that is an isolated singularity of $D/W(L, D)$.

2. Base change

This section closely follows the material from Chapter 3 of [Sch10] where we shift attention to establish a base change identity.

Definition 2.1. Let $G := \mathrm{GL}_2(\mathbb{Q}_p)$ and $G_r := \mathrm{GL}_2(\mathbb{Q}_{p^r})$.

Let σ be the Frobenius lift on G_r.

For $\delta \in G_r$, let $N\delta := \delta\delta^\sigma \cdots \delta^{\sigma^{r-1}}$.

Fact 2.2. *The conjugacy class of* $N\delta$ *always contains an element of* G.

Definition 2.3. For $\gamma \in G$, define the centralizer

$$G_\gamma(R) := \{g \in \mathrm{GL}_2(R) \mid g^{-1}\gamma g = \gamma\},$$

and for $\delta \in G_r$, define the twisted centralizer

$$G_{\delta,\sigma}(R) := \{h \in \mathrm{GL}_2(R \otimes G_r) \mid h^{-1}\gamma h^\sigma = \delta\}.$$

Fact 2.4. *It is known that* $G_{\delta,\sigma}$ *is an inner form of* $G_{N\delta}$.

Definition 2.5. We choose Haar measures on $G_{N\delta}(\mathbb{Q}_p)$ and $G_{\delta,\sigma}(\mathbb{Q}_p)$. Define the orbital integral

$$O_\gamma(f) = \int_{G_\gamma(\mathbb{Q}_p)\backslash G} f(g^{-1}\gamma g)dg$$

for any smooth function f with compact support on G.

Define the twisted orbital integral

$$TO_{\delta,\sigma}(\phi) = \int_{G_{\delta,\sigma}(\mathbb{Q}_p)\backslash G_r} \phi(h^{-1}\delta h^\sigma)dh$$

for any smooth function ϕ with compact support on G_r.

Definition 2.6. The functions $f \in C_c^\infty(G), \phi \in C_c^\infty(G_r)$ have matching (twisted) orbital integrals (or are "associated") if for all semisimple $\gamma \in \mathbb{G}$,

$$O_\gamma(f) = \begin{cases} \pm TO_{\delta,\sigma}(\phi) & \text{if } \gamma \text{ is conjugate to } N\delta \text{ for some } \delta \\ 0 & \text{else.} \end{cases}$$

The sign is always positive unless both $N\delta$ is a central element and δ is not σ-conjugate to a central element.

Remark 2.7. This definition depends on the choice of Haar measures on G and G_r (which we do not yet fix) but does not depend on the choice of Haar measures on $G_{N\delta}(\mathbb{Q}_p)$ and $G_{\delta,\sigma}(\mathbb{Q}_p)$ as long as they are chosen compatibly.

Proposition 2.8. *Let* $\delta \in \mathrm{GL}_2(\mathbb{Z}_{p^r}/p^n\mathbb{Z}_{p^r})$. *Then*

$$G_{N\delta}(\mathbb{Z}/p^n\mathbb{Z}) = \{g \in \mathrm{GL}_2(\mathbb{Z}_p/p^n\mathbb{Z}_p \mid g^{-1}N\delta g = N\delta\}$$

has the same number of elements as

$$G_{\delta,\sigma}(\mathbb{Z}/p^n\mathbb{Z}) = \{h \in \mathrm{GL}_2(\mathbb{Z}_{p^r}/p^n\mathbb{Z}_{p^r} \mid h^{-1}\delta h^\sigma = \delta\}.$$

Furthermore, σ*-conjugacy classes in* $\mathrm{GL}_2(\mathbb{Z}_{p^r}/p^n\mathbb{Z}_{p^r})$ *are mapped bijectively to conjugacy classes in* $\mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$ *via the norm map.*

Corollary 2.9. *Define the principal congruence subgroups*

$$\Gamma(p^n)_{\mathbb{Q}_p^r} := \{g \in \mathrm{GL}_2(Z_{p^r}) \mid g \equiv 1 \pmod{p}^n\}.$$

Let f *be a conjugation-invariant locally integrable function on* $\mathrm{GL}_2(\mathbb{Z}_p)$.

Then the function $\phi := f \circ (N * \cdot)$ *on* $\mathrm{GL}_2(\mathbb{Z}_{p^r})$, *i.e.* $\phi(\delta) = f(N\delta)$, *is locally integrable. Furthermore for all* $\delta \in \mathrm{GL}_2(\mathbb{Z}_{p^r})$,

$$(e_{\Gamma(p^k)_{\mathbb{Q}_{p^r}}} * \phi)(\delta) = (e_{\Gamma(p^k)_{\mathbb{Q}_p}} * f)(N\delta).$$

Proof. Assume that f is locally constant, say invariant by $\Gamma(p^n)_{\mathbb{Q}_{p^r}}$. Then ϕ is also invariant by $\Gamma(p^n)_{\mathbb{Q}_{p^r}}$ and locally integrable. The identity follows from combining Proposition 2.8 for k and n.

The corollary follows in general by approximating f by locally constant functions. □

Definition 2.10 (Shintani)**.** Let π and Π be tempered representations of G and G_r respectively. Π is called a "base-change lift" of π if Π is $\mathrm{Gal}(G_r/G)$-invariant and for all $g \in G_r$ such that the conjugacy class of Ng is regular semisimple,

$$\mathrm{tr}(Ng \mid \pi) := \mathrm{tr}\pi(Ng) = \mathrm{tr}(\Pi(g)I_\sigma) =: \mathrm{tr}((g, \sigma) \mid \Pi)$$

where I_σ is the canonical intertwining operator $\Pi \mapsto \Pi^\sigma$ where $\Pi^\sigma(g) := \Pi(\sigma g)$ (we are basically extending Π to a representation of $G_r \rtimes \mathrm{Gal}(G_r/G)$).

Remark 2.11. These are not actual traces, but an integration *á lá* Weyl integration formula.

Base-change lifts are known to exist by [Lan80] and more generally by [AC89, Theorem 6.2].

Theorem 2.12. *Assume* $f \in \mathcal{Z}(G), \phi \in \mathcal{Z}(G_r)$ *such that for every tempered irreducible smooth representation* π *of* G *with base-change lift* Π*, the scalars* $c_{f,\pi} = c_{\phi,\Pi}$ *by which they act agree.*

Then for any $h \in C_c^\infty(G)$ *and* $h' \in C_c^\infty(G_r)$ *with matching (twisted) orbital integrals,* $f * h$ *and* $\phi * h'$ *also have matching (twisted) orbital integrals.*

Furthermore, $e_{\Gamma(p^n)_{\mathbb{Q}_p}}$ *and* $e_{\Gamma(p^n)_{\mathbb{Q}_{p^r}}}$ *have matching (twisted) orbital integrals.*

Remark 2.13. Here, $c_{f,\pi} = \omega_\pi(f)$ and $c_{\phi,\Pi} = \omega_\Pi(\phi)$.

Proof. Using that h and h' have matching (twisted) orbital integrals, we know $\mathrm{tr}(h \mid \pi) = \mathrm{tr}((h', \sigma)|\Pi)$ if Π is a base-change lift of π (consequence of Weyl integration formula and twisted version by [Lan80]).

Then

$$\mathrm{tr}(f * h \mid \pi) = c_{f,\pi}\mathrm{tr}(h \mid \pi) = c_{\phi,\Pi}\mathrm{tr}((h', \sigma) \mid \Pi) = \mathrm{tr}((\phi * h', \sigma) \mid \Pi).$$

Per [Lan80], we can find a function $f' \in \mathcal{H}(G)$ that has matching (twisted) orbital integrals with $\phi * h'$, so $\mathrm{tr}((\phi * h', \sigma \mid \Pi) = \mathrm{tr}(f' \mid \pi)$. Hence, $\mathrm{tr}(f * h - f' \mid \pi) = 0$ for all tempered irreducible smooth representations π of G. By Kazhdan's density theorem (1986), all regular semi-simple orbital integrals of $f * h - f'$ vanish. This is the only difference between the regular semisimple (twisted) orbital integrals of $f * h$ and $\phi * h'$ so they match. By Clozel (1990), all of their semisimple (twisted) orbital integrals match (not just regular).

The last statement is to check

$$\mathrm{tr}(e_{\Gamma(p^n)_{\mathbb{Q}_p}} \mid \pi) = \mathrm{tr}((e_{\Gamma(p^n)_{\mathbb{Q}_{p^r}}}, \sigma) \mid \Pi),$$

which follows from the same argument and Corollary 2.9 using the restriction of the character of π to $\mathrm{GL}_2(\mathbb{Z}_p$ (characters are locally integrable) as f, $k = n$, and $\delta = 1$. ☐

References

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